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DIFFUSION TO A PARTICLE AT LARGE PÉCLET NUMBERS IN THE CASE OF ARBITRARY AXISYMMETRIC FLOW OVER A VISCOUS FLUID

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An expression for a diffusive flux of matter onto a surface of a solid or liquid particle is obtained for the case of an arbitrary axisymmetric flow-over, containing an arbitrary number of critical lines. This is a generalization of results obtained in [1, 2] where various type constraints were imposed on the field of flow. The formulas obtained make it possible to compute the distribution of the concentration and mass exchange coefficients, from the data on the velocity field near the particle. Formulas for computing the diffusive mass exchange of a rigid spherical particle in a flow with a parabolic velocity profile, and of an ellipsoid of revolution in an uniform translational flow are given.

In the course of the analysis we assume that the concentration of material dissolved in the fluid is constant at a distance from the particle, and that the material is fully absorbed at its surface. In the spherical coordinate system tied to the particle, the equation of convective diffusion and the boundary conditions, have the form

$$v_r \frac{\partial c}{\partial r} + \frac{v_\theta}{r} \frac{\partial c}{\partial \theta} = D\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial c}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial c}{\partial \theta}\right)\right] \qquad (1)$$

$$c\,\big|_{r=R(\theta)} = 0, \quad c\,\big|_{r\to\infty} = c_0$$

$$v_r = \frac{1}{r^2 \sin\theta}\frac{\partial \psi}{\partial \theta}, \quad v_\theta = -\frac{1}{r\sin\theta}\frac{\partial\psi}{\partial r}$$

Here c is concentration, $r = R(\theta)$ is the equation of the particle surface, v_r and v_θ are the fluid velocity components and ψ is the stream function. Assuming that the Péclet number $P = aU \mid D \gg 1$ (D is the coefficient of diffusion, a is the characteristic dimension of the particle and U is the characteristic velocity of flow), we neglect the diffusive transfer of the material along the surface of the particle as compared with its transfer along the normal. In the general case of an axisymmetric laminar flow of a viscous fluid past a particle, the stream function near the surface of the particle can be written in the form

$$\psi_n = (r - R(\theta))^n f_n(\theta), \quad f_n(\theta) = \frac{1}{n}\frac{\partial^n \varphi_n}{\partial r^n}\bigg|_{r=R(\theta)} = \qquad (2)$$

$$-\frac{1}{n}\sin\theta R(\theta)\frac{\partial^{n-1} v_\theta}{\partial r^{n-1}}\bigg|_{r=R(\theta)}$$

Here $n = 1$ corresponds to a drop, and $n = 2$ to a rigid particle. Using the variables ψ, θ, we obtain the following expression in the diffusive boundary layer from (1) and (2):

$$\frac{\partial c}{\partial \theta} = -Dn\,(R^2 + R_\theta'^2)\,|f(\theta)|^{1/n} \sin\theta\, \mathrm{sign}\, f(\theta)\, \frac{\partial}{\partial|\psi|}\,|\psi|^{(n-1)/n}\frac{\partial c}{\partial|\psi|} \qquad (3)$$

$$c\,|_{\psi=0}=0, \quad c\,|_{|\psi|\to\infty}=c_0$$

Here and henceforth we omit the subscript n from the functions ψ and f.

The fluid flow along the trajectories originating at infinity and terminating at some points of the particle surface (these are the points at which the diffusive boundary layer is generated), is enriched with the diffusing material to the maximum of its capacity. (Below we assume that the regions of closed circulation are absent). Therefore, to complete the formulation of the problem in the new variables we must assume that the concentration at the inflow trajectories is equal to that at infinity. We must also remember that this condition is a limiting one and hold holds only $P\to\infty$.

Let it consider in more detail the local geometry of the flow (in the diffusive boundary layer). The zeros of the function $f(\theta)$ determine the critical points and separate the regions in which the stream function is of constant sign. Let us define the angles θ_i^- and θ_i^+ so that $f(\theta_i^-)=0, (f_\theta'(\theta)/\sin\theta)_{\theta=\theta_i^-}<0$ and $f(\theta_i^+)=0$, $(f_\theta'(\theta)/\sin\theta)_{\theta=\theta_i^+}>0$. Then the cones $\theta=\theta_i^-$ will represent the inflow trajectories, and $\theta=\theta_i^+$ the outflow trajectories. The inflow and outflow trajectories must follow each other by virtue of the law of conservation of mass. Since the flow around the particle is axisymmetric, $\theta=0$ and $\theta=\pi$ are zeros of $f(\theta)$. We assume for definiteness, that $\theta_1^-=0$ represents an inflow trajectory. Expanding $f(\theta_1^+ + \theta)$ into a series in θ we find, that the stream function is negative in the region $0<\theta<\theta_1^+$. In the same manner we find the signs of the stream function between any two roots of $f(\theta)$.

Let us consider the region

$$\sigma_i=\begin{cases}0\leqslant\theta\leqslant\theta_1^+, & i=1\\ \theta_{i-1}^+\leqslant\theta\leqslant\theta_i^+, & i>1\end{cases}$$

The additional boundary condition for the concentration in σ_i can be written in the form

$$c_i\,|_{\theta=\theta_i^-}=c_0 \tag{4}$$

We introduce into σ_i the following new variable:

$$t_i=-Dn\,\mathrm{sign}\,f(\theta)\,A_n(\theta,\theta_i^-), \quad A_n(\alpha,\beta)=\int_\beta^\alpha \sin\theta\,(R^2+R_\theta'^2)\,|\,f(\theta)\,|^{1/n}\,d\theta \tag{5}$$

The problem (3), (4) of the distribution of concentration in the diffusive boundary layer (in σ_i) has the form

$$\frac{\partial c_i}{\partial t_i}=\frac{\partial}{\partial|\psi|}\,|\psi|^{(n-1)/n}\,\frac{\partial c_i}{\partial|\psi|} \tag{6}$$

$$c_i\,|_{\psi=0}=0, \quad c_i\,|_{|\psi|\to\infty}=c_0, \quad c_i\,|_{t_i=0}=c_0$$

A solution of the problem (6) is given by

$$c_i=c_0K_n\int_0^{\eta_i}\exp\left[-\left(\frac{2}{n+1}\right)^2\tau^{n+1}\right]d\tau \tag{7}$$

$$\eta_i=\frac{n}{2}\,|\psi|^{1/n}\,t_i^{-1/(n+1)}, \quad K_n=\begin{cases}2/\sqrt{\pi}, & n=1\\ (2/3)^{2/3}\Gamma(4/3), & n=2\end{cases}$$

Formula (7) together with (2) and (5) describes the distribution of concentration around the particle. The diffusive flux on the particle is

$$j_i(\theta)=D\frac{\partial c_i}{\partial n}\bigg|_{r=R(\theta)}=2^{-(n+2)/3}\,nK_nc_0D^{n/(n+1)}\,\times \tag{8}$$

$$|f(\theta)^{1/n} A_n^{-1/(n+1)}(\theta, \theta_i^-)| \frac{\sqrt{R^2(\theta) + R_\theta'^2(\theta)}}{R(\theta)}$$

The thickness of the diffusion boundary layer is given by the formula

$$\delta_i = Dc_0 / j_i$$

When $\theta \to \theta_i^+$ ($f(\theta_i^+) = 0$), the quantity $\delta \to \infty$. Near these angles the thickness of the diffusive boundary layer is not small compared with the characteristic dimension of the particle. Therefore the method used is inapplicable at the vicinity of θ_i^+. From (5) and (8) we see that these regions diminish in size with increasing values of the Péclet number, and their contribution towards the total diffusive flux on the particle ceases to be significant. The total flux on the particle surface is

$$I = \sum_i \int_{\sigma_i} j_i d\sigma = 2\pi \sum_i \int_{\theta_i^+}^{\theta_{i+1}^+} \sin\theta R^2(\theta) j_i(\theta)\, d\theta = \tag{9}$$

$$N_n \sum_i \int_{\theta_i^+}^{\theta_{i+1}^+} \sin\theta R^2(\theta) |f(\theta)^{1/n} A_n^{-1/(n+1)}(\theta, \theta_i^-)| \frac{\sqrt{R^2(\theta) + R_\theta'^2(\theta)}}{R(\theta)} d\theta$$

$$N_n = 2\pi n K_n 2^{-(n+2)/3} c_0 D^{n/(n+1)}$$

We note that an expression for the total diffusive flux on a particle was obtained for the case of an arbitrary three-dimensional flow past a particle in [3], using an auxilliary function which was obtained by solving a first order partial differential equation with coefficients depending on the geometry of the flow near the body.

The formula (9) for the total diffusive flux simplifies considerably when the particle is nearly spherical

$$R(\theta) = a(1 + \lambda\xi(\theta)), \quad \lambda \ll 1 \tag{10}$$

Substituting (10) into (9), expanding in λ and taking (5) into account, we obtain

$$I_0 = N_n \sum_i \int_{\theta_i^+}^{\theta_{i+1}^+} \sin\theta R^2(\theta) |f(\theta)|^{1/n} |A_n(\theta, \theta_i^-)|^{-1/(n+1)} d\theta + O(\lambda^2) = \tag{11}$$

$$N_n \sum_i \int_{\theta_i^+}^{\theta_{i+1}^+} |A_n(\theta, \theta_i^-)|^{-1/(n+1)} A_\theta'(\theta, \theta_i') \, d\theta + O(\lambda^2) =$$

$$N_n \sum_i \left(\int_{\theta_i^+}^{\theta_i^-} + \int_{\theta_i^-}^{\theta_{i+1}^+} \right) = 2\pi(n+1) 2^{-(n+2)/3} c_0 K_n D^{-n/(n+1)} \times$$

$$\sum_{k=1}^{N-1} A_n^{n/(n+1)}(\theta_{k+1}, \theta_k) + O(\lambda^2)$$

where $\theta_k < \theta_{k+1}$ ($k = 1, \ldots, N$) are the roots of $f(\theta)$.

Expressions for the spherical particles were obtained in [2, 4 — 9] for certain particular cases. We shall use the formulas (9) — (11) to investigate two particular cases which complement the results of [2, 4 — 9].

1°. Diffusion to the surface of a rigid spherical particle in an incident flow with a parabolic velocity profile. At large distances from the particle the velocity field has the form

$$\mathbf{v} = {}^1/_2 T \mathbf{e}_x (y^2 + z^2)$$

Here T denotes the curvature of the velocity profile on the axis of symmetry away from the particle and $\mathbf{e}_x$ is the unit vector along the x-axis (in the Cartesian (x, y, z) -coordinate system). The field of flow serves, in this case, as the first approximation in the method of mirror images [10] for a particle in a field of body forces moving along the axis of the tube (Poiseuille flow), with the density of the particle such that its velocity is the same as the axial velocity of the fluid.

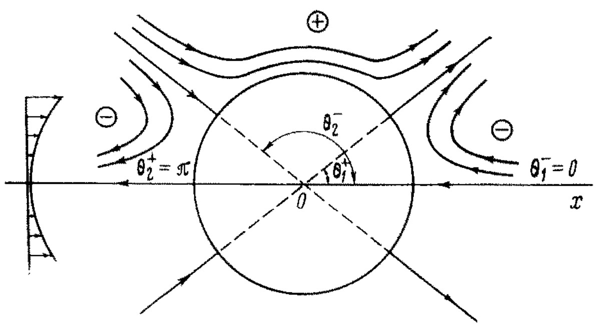

Fig. 1

For the field of flow we shall utilize the results obtained in [11, 12] in the Stokes approximation

$$f(\theta) = -\frac{Ta^2}{4}\sin^2\theta\left(1 - \frac{35}{16}\sin^2\theta\right) \tag{12}$$

From (12) we see that the inflow trajectories are represented by the ray $\theta_1^- = 0$ and the cone $\theta_2^- = \pi - \arcsin(4 / \sqrt{35})$, and the outflow trajectories are $\theta_1^+ = \arcsin(4 / \sqrt{35})$ and $\theta_2^+ = \pi$ (see Fig. 1), $\sigma_1 = \{0 \leqslant \theta_1^+\}$, $\sigma_2 = \{\theta_1^+ \leqslant \theta \leqslant \theta_2^+\}$ (the signs of the stream function are indicated in Fig. 1). The concentration distribution is given by the formulas (2), (5) and (7) with $n = 2$ and $R(\theta) = a$. Using (11) we obtain the following expression for the total flux on the surface of the sphere:

$$I_0 = \frac{3^{1/3}\pi a^2 c_0}{2^{113}\,\Gamma(4/3)} T^{1/3} D^{2/3} \sum_{k=1}^{3} B^{2/3}(\theta_{k+1}, \theta_k)$$

$$B(\alpha, \beta) = \int_\beta^\alpha \sin^2\theta \left|1 - \left(\frac{35}{16}\right)\sin^2\theta\right|^{1/2} d\theta$$

$$B(\theta_2, \theta_1) = B(\theta_4, \theta_3) = \frac{1}{6\sqrt{35}}\left[27E\left(\frac{4}{\sqrt{35}}\right) - 19F\left(\frac{4}{\sqrt{35}}\right)\right]$$

$$B(\theta_3, \theta_2) = \frac{1}{3\sqrt{35}}\left[27E\left(\sqrt{\frac{19}{35}}\right) - 8F\left(\sqrt{\frac{19}{35}}\right)\right]$$

Here F and E are complete elliptic integrals of the first and second kind, respectively. Performing the computations, we obtain

$$I_0 = 6.01 c_0 a^2 D^{2/3} T^{1/3} \tag{13}$$

The same expression was obtained in [9] for the total diffusive flux on a particle in a shear flow with the shear coefficient of $\alpha = 0.0503 aT$.

2°. Diffusion to the surface of a rigid ellipsoid of revolution with a small excentricity in a uniform Stokes flow. We consider the case when $e = a / b - 1$, $| e | \ll 1$, where a and b are the axes of the ellipsoid and a is the symmetry axis oriented along the flow.

The field of flow is obtained by expanding in e the solution given in [13]

$$f(\theta) = {}^3/_4 \sin^2 \theta \, [1 + 4e (\cos^2 \theta - {}^1/_5)]$$

Applying the formula (11) we obtain the following result with the accuracy to within $O(e^2)$:

$$I = I^* (1 - 0.044e)$$

Here I^* denotes the total flux on a sphere of volume equal to that of the ellipsoid of revolution. We see that if the major axis of the ellipsoid is directed with (against) the flow, then its total diffusive flux is smaller (greater) than that for a sphere of the same volume. This is caused by the fact that the ellipsoid has a smaller (larger) velocity gradient at the surface than the sphere.

It should be stressed that the results obtained for a drop ($n = 1$) can also be used in the case of an inviscid or filtration flow past a particle.

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ELASTIC EQUILIBRIUM OF AN ANISOTROPIC CYLINDER WITH LONGITUDINAL CAVITIES SUBJECTED TO AXIAL LOADS

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The stresses and displacements in a long anisotropic cylinder with longitudinal cavities are determined. The problem reduces to seeking an analytic function of a complex variable which is determined in a domain obtained by an affine transformation from the domain of the cylinder cross section. Boundary conditions and a general representation are obtained for the function mentioned.

A long cylinder attenuated by longitudinal cavities, fabricated from a homogeneous linearly-elastic material having a plane of elastic symmetry perpendicular to the cylinder axis at each point is considered. The cylinder is clamped to a rigid mass without tension along the outer surface. Axial tangential forces which do not vary along the cylinder axis are applied to the surfaces of the cavities. Moreover, axial gravitational forces act on the cylinder.

We introduce a rectangular xyz-coordinate system with the z-axis directed downward along the cylinder axis. Let S be a domain occupied by a cross section in the xy-plane, L_0 and L_k are its outer and inner contours ($k = 1, 2, \ldots, N$), γ is the specific gravity of the material, and $\tau_k(s)$ is the intensity of the external forces applied to the k-th cavity.

Following Moskvitin [1] and using the Hooke's law equation in the form written in [2], we find an equation for the axial displacement function

$$A_{44}Dw = A_{55}\frac{\partial^2 w}{\partial x^2} + 2A_{45}\frac{\partial^2 w}{\partial x\,\partial y} + A_{44}\frac{\partial^2 w}{\partial y^2} = -\gamma \tag{1}$$

Here A_{44}, A_{45}, A_{55} are elastic constants of the material. We represent the general solution of (1) as in [2]

$$A_{44}w = w_1 + 2\operatorname{Re}\Phi(z_1)$$

Here w_1 is some particular solution of the inhomogeneous equation $Dw_1 = -\gamma$, and $\Phi(z_1)$ is an analytic function of the auxiliary complex variable $z_1 = x_1 + iy_1$, where

$$x_1 = x + \alpha y, \quad y_1 = \beta y \tag{2}$$